The Cops and Robber Game on Graphs

CS201A Project

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1 Introduction

Cops and Robbers game is a two player game played on a graph which was first introduced in the early 1980s. It is a game played with a set of cops (controlled by one player) trying to capture the robber (controlled by the opposing player). The cops and the robber are restricted vertices, and they can move along the edges to neighboring vertices. The game is also known as vertex-pursuit. The aim of this project to study different classes of graph and theorems which can be used to tell if a cop can win on a given graph and given initial positions of cops and robber.

Let G be a *connected undirected reflexive* graph which doesn't has multiple edges(as it doesn't effect). Two players called cop C and robber R, play a game on G according to the following rules: First C and then R occupy some vertex of G. After that they move alternatively along edges of G. The cop C wins if he succeeds in putting himself on top of R (if he catches R). If the robber R can prevent C from ever catching him, then R wins.

2 Lower Bound

Lower bound is the term used to denote the minimum number of cops required to capture the thief i.e we concentrate on lower limit of $c(G)$.

Graph Characteristics

- If a G is a disjoint union of G_1 and G_2 then $c(G) = c(G_1 + G_2) = c(G_1) + c(G_2)$
- The minimum degree of a graph is known as $\delta(G)$ and maximum degree is known as $\triangle(G)$.
- Corners play important role in characterizing cop-win graphs.
- For paths(P_n),wheels(W_n) and complete graph(K_n) in case of path it's easy to show that c(G)=1 whereas in other cases as we can find a $v \in V$ such that order of v is $n-1$ we can say that $c(G) = 1$. In case of a cycle(C_n) c(G)=2 for $n \geq 4$.

Theorem 2.1. *A finite tree is cop-win , i.e.,* $c(T) = 1$ *if* T *is a tree.*

Proof. Place the cop on an arbitrary vertex of T. On each subsequent round move the cop directly toward the robber along the unique path between them. Eventually the robber will occupy a leaf since T is a finite tree. He will be captured soon after. □

Theorem 2.2. *The cop number of an infinite tree is either 1 or $infinite$. It is exactly* 1 *when the tree is rayless.*

Aigner and Fromme Theorem: If G has girth at least 5, then $c(G) \geq \delta\ (G)$.

Proof. Let G be a graph with $(G) = d$, and suppose $d-1$ cops are playing and C be the set of vertices in G occupied by cops. Let u be a vertex outside of C. Then $N(u) = X \cup Y$, where $X \subset C$ and $Y \subset V(G) - C$. Also,$|X| + |Y| \geq d$. Since, C is a dominating set, so each vertex in Y is adjacent to some vertex in C. As there are neither three *nor* four cycle,no vertex of Y is joined to a vertex of X and no two vertices of Y are adjacent to the same vertex in C . So each vertex of Y is adjacent to a unique vertex of C X. Then $d-1 = |C| \geq |X| + |Y| \geq d$, a contradiction. So C is not a dominating set, hence there is some vertex adjacent to u which is not adjacent to any vertex in C, and the robber may move there in round 0. Suppose in round $t \geq 0$, that the robber occupies some vertex u_t such that no vertex adjacent to u_t is in C. By induction, suppose such a vertex exists for $t-1$. So at time t, the robber is on a vertex u_{t-1} nonadjacent to all vertices in C . As the girth of G is 5, so each cop is adjacent to at most one neighbor of u_{t-1} .As $d(u_{t-1}) \geq d$, so the robber can move to vertex u_t not joined to any vertex in C. □

Frankl Theorem: For a fixed integer $t \geq 1$,if g has girth atleast $8t - 3$ and $\delta(G) > d$, then $C(G) > d^t$.

3 Upper Bound

Not many good upper bounds are known on cop number. For example, $c(G) \leq \gamma(G)$. This is not a good upper bound. Let us consider a path. $c(P_n) = 1$whereas $\gamma(Pn) = [n/3]$.

The best known result on upper bounds is from Lu,Peng

$$c(G) = O\Big(\frac{n}{2^{(1-o(1))\sqrt{\log n}}}\Big)$$

However, our focus will be on Frankl's formula.

Theorem 3.1. *If order of G in n , then*

$$c(G) \leq O\left(n\frac{\log\log n}{\log n}\right)$$

where c(G) is cop-number of G.

We will be learning few things along side.

For a fixed integer $k \geq 1$, an induced subgraph H of G is k-guardable if, after finitely many moves, k cops can move only in the vertices of H in such a way that if the robber moves into H at round t, then he will be captured at round $t+1$. For example, a clique or a closed neighbour set in a graph are 1-guardable, and G is $\gamma(G)$guardable.

A path P in G is isometric if all vertices u and v of P,

$$d_P(u, v) = d_G(u, v)$$

Theorem 3.2. *An isometric path is 1-guardable.*

Proof. Let $P = \{v_0, v_1, v_2, ..., v_k\}$ be an isometric path in a graph G, and let

$$D_i = \{x \in V(G) : d(x, v_0) = i\}$$

Since P is an isometric path, it follows that $v_i \in D_i$ for $i = 0, 1, ...k$.

The cop,restricted to P, plays as if the robber is on v_j when the robber is on some vertex D_j, $j = 0, 1, 2, 3..., k-1$, and on v_k when $j \geq k$. We will call this the robber's image. Say, the robber is on D_j, then he can move to vertices D_{j-1},D_j and D_{j+1}, so his image can only move from v_j to v_{j-1} or v_{j+1} or remain at v_j. Start the cop on v_0. As far as the game between the cop and the image is concerned, the game is won by the cop(as the game is played on a path). After the image has been caught, the actual robber can still move in G, but the image moves on the adjacent vertex on P or is stationary. So the cop moves to recapture the image. Suppose, now the robber tries to enter P. He is at some vertex j. If $j < k$, then its image and the cop are on v_j. The robber can move to one of v_{j-1}, v_j or v_{j+1}. In each case, cop captures the robber. Now, suppose,$j \geq k$, so, its image and the cop are on v_k . Now the only possible moves are to v_k or v_{k-1} and hence the robber is captured on the very next move. □

Theorem 3.3. *(Moore bound)Let G be a graph of order n, with maximum degree $\triangle > 2$ and diameter D. Then*

$$n \leq 1 + \triangle\Big(\frac{(\triangle - 1)^D - 1}{\triangle - 2}\Big)$$

Proof. For a fixed vertex u, and an integer $1 \leq i \leq D$, define $N_i(u)$ to be the set of vertices of distance i to u clearly,

$$\cup_{i=0}^{D} N_i(u) = V(G)$$

The vertex u contributes one to $|N(u)|$, and N_1 contributes at most $\triangle$. By induction it can be proved that for $2 \leq i \leq D, Ni(u)$ contributes at most $\triangle(\triangle - 1)^{i-1}$ to $|N_i(u)|$ □

Proof of theorem 3.1

We know that each closed neighbour set of a vertex u is 1-guardable. By theorem 3.2, we also know that each isometric path of length D is also 1-guardable.

By Moore bound we can conclude that

$$n - O(\triangle^D)$$

And according to Moore bound, $\triangle$ and D cannot be less than $O\Big(\frac{\log n}{\log \log n}\Big)$

There is a subset X consisting of either a closed neighbour set or an isometric path of order at least $\Big(\frac{\log n}{\log \log n}\Big)$

Now, we can remove X from G to form graph G''. Though graph G'' may be disconnected, the robber is restricted to a connected component of G'. Therefore,

$$c(G) \leq c(G') + 1$$

as X is 1-guardable. Now with the help of induction on the above equation

$$c(n) \leq c\Big(\frac{n}{2}\Big) + \frac{n/2}{\frac{\log n}{\log \log n}}$$

$$= O\Big(n \frac{\log \log n}{\log n}\Big)$$

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4 Cops, Robbers, and Retracts

The concept on retract play a vital role in understanding the game. In some cases it can be turn out to be very useful to tell whether a graph is cop-win or not. Let us first define retract.

Let G be any graph and H be an induced subgraph of H. We say H is a *retract* of G if there is a *homomorphism* f from G to H so that $f(x) = x \; \forall x \in V(H)$; that is f is identity on H. The map f is caled *retraction.* Distance between pair of vetrices do not increases in the image.

We will state following theorem

Theorem 4.1. *If H is a retract of G, then $c(H) \leq c(G)$.*

Proof. Let G be a graph and H be its retract. Let $c(G) = k$ for some k. We will play two games, one in G and one in H such that when C moves from vertex u to v in G, then $f(C)$ moves from $f(u)$ to $f(v)$ in H. Let the cops play in G with R restricted to H. G is k-cop-win, so at some point the cops will be about to win in G. Then R and each of its neighbors in H (and its neighbors in G - H = v) are adjacent to some cop. Under the retraction, the edge RC becomes $Rf(C)$, and vC becomes $vf(C)$. So $N[R] \subseteq N[f(C)]$ in H, and the robber loses in H in the next round. So $c(H) \leq k$. □

From above theorem we have the following corollary

Corollary 4.1.1. *If G is a cop-win, then so is each retract of G.*

This corollary is very useful when checking if a given graph G is cop-win or not. If G is cop-win and has a retract then the retract is again a cop-win graph. We can add new vertices and edges to the graph to form new graph G', such that G is retract of G', which can be easily verified as a cop-win graph or not.

5 Characterization

A *graph* is *dismantlable* if some sequence of deleting *corners* would result in *graph* K_1. An ordering $v_1...v_n$ of the vertices of a graph is a cop-win ordering if for each $i < n$ there is a $j > i$ such that $N_i[v_i] \subseteq N_j[v_j]$.

We will state following results

Lemma 5.1. *If G is a $cop - win$ graph, then G contains atleast 1 corner.*

Proof. Consider second to last move of the cop. The robber could pass, so C must be joined to R. Otherwise robber could move to neighboring vertex, so C must be joined to each vertex of R. Therefore, by definition of corner, the second to last position of robber is corner. □

Theorem 5.2. *A graph is $cop - win$ if and only if it is dismantlable*

Proof. Consider G is a $cop - win$ $graph$. In proving G is dismantlable we will use mathematical induction on the order of G. Base case is trivial as $G \cong K_1$. By above proved lemma each $cop - win$ $graph$ has a corner u dominated by some other vertex. Now, form $G - u$ and note that it is retract of G. We know that if G is $cop - win$ then each retract of G is also $cop - win$. So, $G - u$ is $cop - win$ whose order is less than order of G, which implies that $G - u$ is dismantlable. Therefore, G is dismantlable.
Now, consider G is dismantlable. We use induction on order of G to prove that G is cop-win. In Base case, $G \cong K_1$, which is cop-win. Consider G is dismantlable of order $n + 1$. Then, G contains some *corner* u with $u \to v$, for some v so that $G - u$ is dismantlable. By induction hypothesis $G - u$ is cop-win and has order n.To show that G is cop-win, we use shadow strategy. Consider cop playing in $G - u$ with his winning strategy there. Whenever, R moves to u, C moves as though R moves to v. We think all vertices of $G - u$ as image under retraction f, which maps u to v and fix all others. Now, C eventually captures image $f(R)$ with his winning strategy on $G - u$. Now, either $f(R) = R$ in which case robber is captured or R is on u with cop on v.So, in the next move cop wins as u is dominated by v. □

From the above results we can derive a very important result

Cop-win(or no-backtrack) strategy Assume that $[n]$ is a cop-win ordering of G, and for $1 \leq i \leq n$ define

$$G_i = G\lceil\{n, n-1, \cdots, i\}$$

Let, $f_i : G_i \to G_{i+1}$ be the retraction map from G_i to G_{i+1} mapping i onto vertex that covers i in G_i. Define F_1 to be identity mapping on G. For $2 \leq i \leq n$ define

$$F_i = f_{i-1} \circ \cdots \circ f_2 \circ f_1.$$

In other words, F_i is the mapping formed by iteratively retracting corners $1, 2, \cdots, i - 1$. For all i, as the f_i are retractions, $F_i(x)$ and $F_{i+1}(x)$ are either equal or joined. If the robber is on vertex x in G, then we think $F_i(x)$ as robbers shadow on G_i. Now, the cop begins on vertex n (G_n), which is the shadow of robbers position under F_n. Suppose the robber is on u and the cop is occupying the shadow of robber in G_i equaling $F_i(u)$. If the robber moves to v, then the cop moves onto the image $F_{i-1}(v)$ of R in the larger graph G_{i-1}.

6 Refrences

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